

# Negative Power Spectra in Quantum Field Theory

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We consider the spatial power spectra associated with fluctuations of quadratic operators in field theory, such as quantum stress tensor components. We show that the power spectrum can be negative, in contrast to most fluctuation phenomena where the Wiener-Khinchine theorem requires a positive power spectrum. We show why the usual argument for positivity fails in this case, and discuss the physical interpretation of negative power spectra. Possible applications to cosmology are discussed.

PACS numbers: 03.70.+k, 04.62.+v, 05.40.-a

The well-known Wiener-Khinchine [1, 2] theorem states that the Fourier transform of a correlation function is a power spectrum. A corollary of this theorem is that the power spectrum can normally be written as the expectation value of a squared quantity, and hence must be positive. The theorem is most commonly formulated for temporal Fourier transforms. (For a review, see for example, Ref. [3].) However, it may also be formulated for spatial Fourier transforms, which will be our focus. The purpose of this letter is to note that there is a loophole in the proof of the corollary, which allows for the possibility of negative power spectra. Furthermore, we show that such negative spectra can arise in the case of the fluctuations of quadratic operators in quantum field theory, such as the energy density operator.

First, let us recall the more familiar case where the power spectrum must be non-negative. Let  $F(t, \mathbf{x})$  be a fluctuating quantity, e.g., a Hermitian quantum operator in flat spacetime, and let the associated correlation function be

$$C(t - t', \mathbf{x} - \mathbf{x}') = \langle F(t, \mathbf{x}) F(t', \mathbf{x}') \rangle. \quad (1)$$

Define  $\hat{F}(t, \mathbf{k})$  to be a spatial Fourier transform

$$\hat{F}(t, \mathbf{k}) = \frac{1}{(2\pi)^3} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} F(t, \mathbf{x}), \quad (2)$$

and let its correlation function be

$$C(t - t', \mathbf{k}, \mathbf{k}') = \langle \hat{F}(t, \mathbf{k}) \hat{F}(t', \mathbf{k}') \rangle. \quad (3)$$

If we use insert Eq. (2) into Eq. (3), and change integration variables to  $\mathbf{u} = \mathbf{x} - \mathbf{x}'$  and  $\mathbf{v} = \mathbf{x} + \mathbf{x}'$ , we find

$$C(t - t', \mathbf{k}, \mathbf{k}') = \frac{1}{4(2\pi)^3} \delta(\mathbf{k} - \mathbf{k}') \int d^3u e^{i\mathbf{k}\cdot\mathbf{u}} C(t - t', \mathbf{u}). \quad (4)$$

Hence

$$C(0, \mathbf{k}, \mathbf{k}') = P(k) \delta(\mathbf{k} - \mathbf{k}'), \quad (5)$$

where the power spectrum  $P(k)$  is defined by

$$P(k) = \frac{1}{(2\pi)^3} \int d^3u e^{i\mathbf{k}\cdot\mathbf{u}} C(0, \mathbf{u}). \quad (6)$$

If  $C(0, \mathbf{k}, \mathbf{k}')$  exists, then it is non-negative, as it is the expectation value of  $\hat{F}^2(t, \mathbf{k})$ , and as a consequence  $P(k) \geq 0$ . This is a form of the Wiener-Khinchine theorem for spatial Fourier transforms.

A possible loophole can arise if  $C(0, \mathbf{k}, \mathbf{k}')$  is not well-defined. A simple example can illustrate this possibility. Consider the operator

$$F(t, \mathbf{x}) =: \varphi^2 :, \quad (7)$$

where  $\varphi$  is a massless free scalar field in four-dimensional Minkowski spacetime. The vacuum correlation function in coordinate space may be found from Wick's theorem to be

$$C(\tau, r) = \frac{1}{8\pi^4 (r^2 - \tau^2)^2}, \quad (8)$$

where  $\tau = t - t'$ . The spatial Fourier transform is

$$\hat{C}(\tau, k) = \frac{1}{(2\pi)^3} \int d^3x e^{i\mathbf{k}\cdot\mathbf{x}} C(\tau, r) = -\frac{\sin(k\tau)}{64\pi^5 \tau}. \quad (9)$$

This result may be verified either by direct evaluation of the above integral using contour integration methods, or by checking that

$$\begin{aligned} C(\tau, r) &= \int d^3k e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{C}(\tau, k) \\ &= -\frac{1}{16\pi^4 r \tau} \lim_{\alpha \rightarrow 0} \int_0^\infty dk k \sin(kr) \sin(k\tau) e^{-\alpha k}. \end{aligned} \quad (10)$$

The associated power spectrum is

$$P(k) = \hat{C}(0, k) = -\frac{k}{64\pi^5}, \quad (11)$$

which is negative. Clearly, the positivity argument given above does not hold in this case, and the reason must be that  $\mathcal{C}(0, \mathbf{k}, \mathbf{k}')$  does not exist as a well-defined quantity.

The explicit form for  $F(\mathbf{x}, t)$  in terms of creation and annihilation operators is

$$F(t, \mathbf{x}) = \sum_{\mathbf{q}, \mathbf{q}'} \frac{1}{2V\sqrt{\omega\omega'}} \left\{ e^{i[(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}-(\omega-\omega')t]} a_{\mathbf{q}'}^\dagger a_{\mathbf{q}} + e^{i[(\mathbf{q}+\mathbf{q}')\cdot\mathbf{x}-(\omega+\omega')t]} a_{\mathbf{q}} a_{\mathbf{q}'} + H.c. \right\}. \quad (12)$$

Here we use box normalization in a volume  $V$ , H.c. denotes the Hermitian conjugate,  $\omega = |\mathbf{q}|$ , and  $\omega' = |\mathbf{q}'|$ . The Fourier transformed operator is

$$\hat{F}(t, \mathbf{k}) = \frac{1}{2(2\pi)^3} \sum_{\mathbf{q}} \frac{1}{\sqrt{\omega\omega_1}} \left[ e^{i(\omega_1-\omega)t} a_{\mathbf{q}+\mathbf{k}}^\dagger a_{\mathbf{q}} + e^{-i(\omega+\omega_1)t} a_{-\mathbf{q}-\mathbf{k}} a_{\mathbf{q}} + H.c. \right], \quad (13)$$

where  $\omega_1 = |\mathbf{q} + \mathbf{k}|$ .

The correlation function for  $\hat{F}(\mathbf{k}, t)$  can be shown to be

$$\mathcal{C}(\tau, \mathbf{k}, \mathbf{k}') = \frac{\delta_{\mathbf{k}, \mathbf{k}'}}{2(2\pi)^6} \sum_{\mathbf{q}} \frac{1}{\omega\omega_1} \cos[(\omega_1 - \omega)\tau], \quad (14)$$

where  $k = |\mathbf{k}|$ . If we take the limit  $\tau \rightarrow 0$ , this quantity is formally positive, but divergent and hence undefined. This is the positive quantity that would appear in the Wiener-Khinchine theorem and implied a positive power spectrum had it been well-defined. If we first take the large  $V$  limit and perform the integration on  $\mathbf{q}$ , then the result is

$$\mathcal{C}(\tau, \mathbf{k}, \mathbf{k}') = -\delta(\mathbf{k} - \mathbf{k}') \frac{\sin(k\tau)}{64\pi^5 \tau}, \quad (15)$$

where we have used  $V \delta_{\mathbf{k}, \mathbf{k}'} / (2\pi)^3 \rightarrow \delta(\mathbf{k} - \mathbf{k}')$ . Now we can take the  $\tau \rightarrow 0$  limit, and obtain the negative power spectrum in Eq. (11).

A more physically interesting example arises in the case of the fluctuations of quantum stress tensors, such as the energy density of the electromagnetic field. Let  $\rho(t, \mathbf{x})$  be the normal-ordered energy density operator, and its vacuum correlation function be

$$C_0(\tau, r) = \langle \rho(t, \mathbf{x}) \rho(t', \mathbf{x}') \rangle = \frac{(\tau^2 + 3r^2)(r^2 + 3\tau^2)}{\pi^4(r^2 - \tau^2)^6}, \quad (16)$$

The explicit form of its spatial Fourier transform is

$$\hat{C}_0(\tau, k) = -\frac{k^4 \sin(k\tau)}{960\pi^5 \tau}. \quad (17)$$

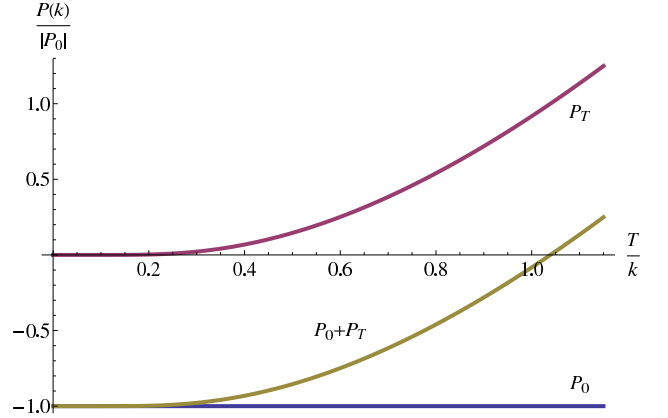


FIG. 1: The vacuum and thermal power spectra,  $P_0$  and  $P_T$ , for the electromagnetic field are plotted as functions of temperature. Here we use units in which Boltzmann's constant is unity,  $k_B = 1$ . The total power spectrum  $P_0 + P_T$ , is also plotted, and seen to be positive for  $T > 1.04k$ .

Again, this result may be confirmed either by contour integration, or by checking that the inverse Fourier transform of  $\hat{C}_0(\tau, k)$  is indeed  $C_0(\tau, r)$ , as illustrated in Eq. (10). The power spectrum is

$$P_0(k) = \hat{C}_0(0, k) = -\frac{k^5}{960\pi^5}, \quad (18)$$

which is again negative.

This leads us to the question of the physical interpretation of negative power spectra. One possibility is that the vacuum fluctuation contribution is a sub-dominant contribution which reduces the magnitude of a net positive power spectrum. This possibility can be illustrated by energy density fluctuations at finite temperature. The electromagnetic energy density correlation function at finite temperature can be obtained from Eq. (16) as an image sum over imaginary time by replacing  $\tau$  by  $\tau + in\beta$ , and summing over all integers  $n$ . Here  $\beta = 1/(k_B T)$ , where  $k_B$  is Boltzmann's constant, and  $T$  is the temperature. The temperature dependent part of the correlation function can be written as

$$C_T(0, r) = 2 \sum_{n=1}^{\infty} C_0(in\beta, r) = \frac{2}{\pi^4} \sum_{n=1}^{\infty} \frac{(3r^2 - n^2\beta^2)(r^2 - 3n^2\beta^2)}{(r^2 + n^2\beta^2)^6}, \quad (19)$$

when  $\Delta t = 0$ . The spatial Fourier transform of this expression is  $P_T(k)$ , the temperature dependent part of the power spectrum, which is found to be

$$P_T(k) = -\frac{k^4}{480\pi^5 \beta} \ln(1 - e^{-\beta k}). \quad (20)$$

Note that  $P_T(k) > 0$  for all  $k$ , as may be seen in Fig. 1. We also see from this figure that the total power,  $P_0 + P_T$ ,

is positive for  $T > 1.04k/k_B$ . Thus at high temperature, the negative contribution from the vacuum fluctuations reduces the magnitude of the net power. At lower temperatures, the vacuum term dominates, and the total power is negative.

Consequently, we also need a physical interpretation of the case of net negative power. Note that negative power spectra are always associated with coordinate space correlation functions which are singular at coincident points, and hence cannot represent the expectation value of a meaningful squared quantity. However, the correlation function at distinct points is meaningful, and can have either sign. If  $C(t - t', \mathbf{x} - \mathbf{x}') > 0$ , the fluctuations at  $(t, \mathbf{x})$  are correlated with those at  $(t', \mathbf{x}')$ , and conversely if  $C(t - t', \mathbf{x} - \mathbf{x}') < 0$ , they are anticorrelated. Changing the sign of the power spectrum,  $P(k) \rightarrow -P(k)$ , changes the sign of  $C(t - t', \mathbf{x} - \mathbf{x}')$ , and hence interchanges correlations and anticorrelations. So long as  $C(0, 0)$  is undefined, both situations are logical possibilities. Consider the contribution of a finite bandwidth to a spatial correlation function and define

$$C_{\Delta k}(0, r) = \int_{k_0 \leq k \leq k_1} d^3k e^{-i\mathbf{k} \cdot \mathbf{x}} P(k). \quad (21)$$

This quantity would describe the spatial correlations in a situation where fluctuations with  $k < k_0$  or  $k > k_1$  have essentially been filtered out. It will typically be a quasi-oscillatory function in space. Changing the sign of the power spectrum in this interval interchanges the minima and maxima of  $C_{\Delta k}(0, r)$ , interchanging correlations and anticorrelations.

In general, it is not  $C(t - t', \mathbf{x} - \mathbf{x}')$  itself, but rather integrals of the correlation function over finite spacetime regions which are observable. Let  $S_1(x)$  and  $S_2(x')$  be test functions which describes the effect of some measuring apparatus. The correlation function for the outcomes of the measurements described by  $S_1$  and  $S_2$  is

$$K = \int d^4x S_1(x) \int d^4x' S_2(x') C(t - t', \mathbf{x} - \mathbf{x}'). \quad (22)$$

Even though the function  $C$  is singular at coincident points, it is well-defined as a distribution, so  $K$  is finite. Consider the case of the vacuum energy density of the electromagnetic field, where  $C$  is given by Eq. (16). This correlation function may be expressed as a total derivative [4]:

$$C_0(\tau, r) = -\frac{1}{3840\pi^4} \nabla^2 \square \nabla'^2 \square' \ln^2[(x - x')^2/\ell^2], \quad (23)$$

where  $\ell$  is an arbitrary length. Here  $\nabla^2$  and  $\square$  denote the Laplacian and d'Alembertian operators, respectively, in  $x$ , and  $\nabla'^2$  and  $\square'$  the corresponding operators in  $x'$ . The value of  $C_0(\tau, r)$  is unchanged if  $\ell$  changes. We next integrate by parts in the expression for  $K$  and assume that the surface terms vanish, which will be the case if  $S_1$

and  $S_2$  vanish sufficiently rapidly at infinity. The result is

$$K = \int d^4x \nabla^2 \square S_1(x) \times \int d^4x' \nabla'^2 \square' S_2(x') \ln^2[(x - x')^2/\ell^2]. \quad (24)$$

This expression contains only an integrable, logarithmic singularity and is hence finite. [An explicit example for  $K$  is illustrated in Fig. 5 in Ref. [4].] The key point is that the singular nature of the coordinate space correlation function, which always accompanies negative power spectra, does not prevent quantities such as  $K$  from being well-defined.

Radiation pressure fluctuations on a mirror can be computed as integrals of a quantum stress tensor correlation function [5], yielding a result which may also be derived by an alternative approach based on photon number fluctuations [6]. This is an illustration of how properly defined integrals of singular correlation functions are physically meaningful.

Quantum stress tensor fluctuations can potentially contribute to the primordial density fluctuation spectra in inflationary models [7, 8]. Here the observable quantities involve time integrals of the energy density correlation function. Even though the integrands are singular, the integrals are finite. In a model with a single scalar inflaton field, the contribution of electromagnetic energy density fluctuations to the density fluctuation power spectrum can be written as [See Eq. (88) of Ref. [8].]

$$P_{\delta\rho}(k) = \frac{\ell_P H S^2}{30\pi^2} \left( -S + \frac{4\pi H}{5k} \right), \quad (25)$$

where  $\ell_P$  is the Planck length,  $H$  is the Hubble parameter during inflation, and  $S$  is the scale factor change during inflation. Note that the first term is negative. In the models discussed in Ref. [8], this term is dropped, as it corresponds to a delta-function term in the coordinate space correlation function, and is hence not observable in measurements made in disjoint regions. However, if there is any process which has the effect of filtering this negative power spectrum, it would no longer be a delta-function and hence could become observable. Classical, non-linear evolution of the density perturbations is one possible filtering mechanism. Thus the appearance of this term illustrates the possibility of negative power spectra in cosmology. The appearance of a negative power spectrum of tensor perturbations in inflationary cosmology will be discussed in a separate publication [9].

We have focused on fluctuations in space, and power spectra defined by spatial Fourier transforms. As noted earlier, the Wiener-Khinchine theorem can also be formulated in terms of temporal Fourier transforms and an analogous power spectrum  $P(\omega)$  can be defined. If the

correlation function in time is finite in the coincidence limit, then  $P(\omega) \geq 0$ . In the case of examples studied earlier, the coincidence limit does not exist, but nonetheless one finds a positive power spectrum. For example, for the case of the vacuum fluctuations of the electromagnetic energy density, the power spectrum for temporal fluctuations is

$$P(\omega) = \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} C_0(\tau, 0) = \frac{\omega^7}{560\pi^2} > 0. \quad (26)$$

It is not known whether there are examples of negative temporal power spectra.

In this letter, we have illustrated how the fluctuations of quadratic operators in field theory can produce negative power spectra for spatial fluctuations. A negative contribution can have the effect of decreasing a net positive spectrum. However, it is also possible for the net spectrum to be negative, as in the case of vacuum energy density fluctuations of the quantized electromagnetic field. Negative power spectra have the opposite correlation-anticorrelation behavior as does a positive spectrum with the same functional form.

We would like to thank Shun-Pei Miao, Kin-Wang Ng, Richard Woodard, and the participants of the 14th and 15th Peyresq workshops for valuable discussions. This

work was supported in part by National Science Foundation Grant PHY-0855360 and by the National Science Council, Taiwan, ROC under the Grant NSC99-2112-M-031-002-MY3.

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